On CT and CSA Groups and Related Ideas

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Abstract

A group is G **commutative transitive** or CT if commuting is transitive on nontrivial elements. A group G is CSA or **conjugately separated abelian** if maximal abelian subgroups are malnormal. These concepts have played a prominent role in the studies of fully residually free groups, limit groups and dicriminating groups. They were especially important in the solution to the Tarski problems. CSA always implies CT however the class of CSA groups is a proper subclass of the class of CT groups. For limit groups and finitely generated elementary free groups they are equivalent. In this paper we examine the relationship between the two concepts. In particular we show that a finite CSA group must be abelian. If G is CT then we prove that G is not CSA if and only if G contains a nonabelian subgroup G_0 which contains a nontrivial abelian subgroup H that is normal in G_0. For K a field the group $PSL(2,K)$ is never CSA but is CT if char$(K) = 2$ and for fields K of characteristic 0 where -1 is not a sum of two squares in K. For characteristic p, for an odd prime p, $PSL(2,K)$ is never CT. Infinite CT groups G with a composition series and having no nontrivial normal abelian subgroup must be monolithic with monolith a simple nonabalian CT group. Further if a group G is monolithic with monolith N isomorphic to $PSL(2,K)$ for a field K of characteristic 2 and G is CT then $G \cong N$.

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1 Introduction

A group G is **commutative transitive**, which we will abbreviate by CT, if commutativity is transitive on nonidentity elements. Commutative transitivity is a simple idea that suprisingly has had a wide-ranging impact on many areas of algebra in general and group theory in particular. Of special interest is the important role that commutative transitivity has played in the solution of the celebrated Tarski conjectures. The paper [FR] contains a great deal of information about CT groups in general.

A group G is CSA or **conjugately separated abelian** if maximal abelian subgroups are malnormal (see section 2). CSA implies CT (see section 2) but the class of CSA groups is a proper subclass of the class of CT groups. These two concepts and their relationship have played a major role in the proof of the celebrated Tarski problems (see section 2). A result of Gaglione and Spellman [GS] and independently Remeslennikov [Re] showed that for nonabelian residually free groups, being CT is equivalent to having the same universal theory as a nonabelian free group (see section 2). This result was one of the initial important steps in the solution of the Tarski conjectures (see Section 2).

The term **commutative transitive** was coined in [F] relative to free groups and Fuchsian groups yet the concept appeared in the literature substantially earlier. In some papers a CT group is referred to as **centralizer abelian** or CA-group since being CT is easily shown to be equivalent to having all centralizers of nontrivial elements abelian.

Finite CT groups were studied originally by Weisner [W] in 1925. He proved that finite CT groups are either solvable or simple. However there was a mistake in his proof. Yu-Fen Wu in 1997 [Wu] corrected the mistake and reproved Weisner's result. She also proved that a finite solvable CT group is the semidirect product of its Fitting subgroup F, which must be abelian. by a fixed point free group of automorphisms of F. Earlier Suzuki [Su], in 1957, using character theory proved that every finite nonabelian simple CT group is isomorphic to some $PSL(2, 2^f), f \geq 2$.

In this paper, we examine the relationship between the two concepts, CT and CSA. In the next section we review some important material on CT and CSA groups. As mentioned above, CSA implies CT however there do exist groups, both finite and infinite which are CT but not CSA. For limit groups, however, as well as elementary free groups and some related groups, the two concepts are equivalent. We provide a quick proof of this for limit groups in section 3.

We next consider finite CSA groups and prove, using the results of Wu, that a finite CSA must be abelian. Hence a finite CT group that is not simple and not CSA must have a nontrivial abelian normal subgroup. For infinite groups we prove that a group G that has a composition series and is CT but not CSA either contains a nontrivial normal abelian

subgroup or is monolithic with monolith isomorphic to $PSL(2,K)$ for a field of characteristic 2. Here we use the fact that CSA is given by a set of universal sentences and hence is true if and only if it is true in subgroups. The equivalence of CT and CSA carries over to the class of $B\mathcal{X}$-groups introduced by Ciobanu, Fine and Rosenberger [CFR].

2 Basic Material on CT and CSA Groups

A group G is **commutative transitive** or CT if commutativity is transitive on nontrivial elements. That is

$$[x,y]=1 \text{ and } [y,z]=1 \implies [x,z]=1$$

provided x,y,z are nontrivial.

It is straightforward that being commutative transitive is equivalent to the property that the centralizer of every nontrivial element is abelian. For this reason CT groups are sometimes called CA groups or centralizer-abelian groups.

G is CT if and only if it satisfies the universal sentence

$$\forall x,y,z(((y\neq 1)\wedge(xy=yx)\wedge(yz=zy))\to(xz=zx)).$$

It is also clear is that if $Z(G)\neq\{1\}$ and G is CT then G is abelian.

It is clear (and follows directly from the fact that CT is captured by a universal sentence) that subgroups of CT groups are CT.

Lemma 2.1. *If G is CT then any subgroup of G is also CT.*

Harrison [Ha] first published the following lemma that ties together the CT property with abelian centralizers.

Lemma 2.2 (Ha)**.** *Let G be a group. The following three statements are pairwise equivalent.*

(i) G is commutative transitive.

(ii) The centralizer $C(x)$ of every nontrivial element in $x\in G$ is abelian.

(iii) Every pair of distinct maximal abelian subgroups in G has trivial intersection.

Finite CT groups were studied originally by Weisner [W] in 1925 who proved that finite CT groups are either solvable or simple. However there was a mistake in his proof that was corrected by Yu-Fen Wu in 1997 [Wu]. She further proved that a finite solvable CT group is the semidirect product of its Fitting subgroup F, which must be abelian, by a fixed point

free group of automorphisms of F. Suzuki [Su] in 1957 using character theory proved that every finite nonabelian simple CT group is isomorphic to some $PSL(2, 2^f), f \geq 2$.

Wu further developed a complete structure theory for locally finite CT groups analogous to that of finite CT groups. Her main result that is important for this paper is:

Theorem 2.1 (Wu)**.** *An insolvable locally finite group is CT if and only if $G \cong PSL(2, F)$ for some locally finite field F of characteristic 2 with $|F| \geq 4$,*

Corollary 2.1. *A finite CT group is either solvable or simple and isomorphic to $PSL(2, K)$ for some finite field of characteristic 2.*

There are many examples of classes of CT groups.

(1) Free groups: It is well known that centralizers of elements in nonabelian free groups are all cyclic (see [MKS] or [LS]). This is usually expressed by saying that two elements commute only if they are powers of a common element.

(2) Torsion-free hyperbolic groups: Again here centralizers are cyclic (see [FGMS1]).

(3) Free solvable groups: This was proved by Wu in the paper cited above [Wu]. She also proved that there are solvable CT groups of any derived length.

(4) Free metabelian groups: Again see [Wu].

(5) $PSL(2, \mathbb{R})$ where $\mathbb{R}$ is the real field: (see the next section)

Further from the observation that $PSL(2, \mathbb{R})$ is CT and the obvious fact that all subgroups of CT groups are also CT it follows that all **Fuchsian groups** are CT. In particular any orientable surface group S_g with $g \geq 2$ must be CT. This is important relative to the tie between CT groups and residually free groups.

We note that $PSL(2, \mathbb{C})$ and more generally $PSL(2, K)$ with K an algebraically closed field of characteristic 0 is never CT (see next section).

(6) $PSL(2, K)$ where K is a field with $char(K) = 2$: We will prove this in the next section using model theory. This result will be important in examining the relationship between CT and CSA.

We note that if K is any field of characteristic $p \neq 2$ then $PSL(2, K)$ is **not** CT. This follows from Wu's result on finite CT groups and we give a proof in the next section.

(7) Tarski groups: These are simple groups where every proper subgroup is cyclic hence centralizers are cyclic.

(8) One-relator groups with torsion: This is a consequence of results of Pride on two-generator one-relator groups coupled with earlier results of B.B. Newman and Ree and Mendelsohn (see [FR] and the references there).

Centralizers of elements and abelian subgroups in general are relatively well understood relative to group amalgams, that is free products with amalgamation and HNN groups. Using these ideas, commutative transitivity was studied relative to certain group amalgams by Levin and Rosenberger [LR].

Theorem 2.2 (LR)**.** *If G_1, G_2 are CT groups and H is malnormal and proper in both then the amalgamated product $G_1 \star_H G_2$ is also CT.*

They note that the malnormality condition cannot be relaxed. For example if the values $p, q > 1$ then the amalgamated product

$$\langle a \rangle \star_K \langle b, c; [b, c] = 1 \rangle$$

where $K = \langle a^p \rangle = \langle b^q \rangle$ is not CT.

The situation for HNN extensions of CT groups is not as general but can be carried through for extensions of centralizers of abelian malnormal subgroups.

Theorem 2.3. *Let B be a CT group and K an abelian malnormal subgroup of B. Then the HNN extension*

$$B_1 = \langle t, B; rel(B), t^{-1}kt = k \text{ for all } k \in K \rangle$$

is a CT-group.

Myasnikov and Remeslennikov in their study of fully residually free groups introduced the concept of a CSA group (conjugately separated abelian group). Recall that if G is a group and H a subgroup of G then H is **malnormal** in G or **conjugately separated** in G provided $g^{-1}Hg \cap H = 1$ unless $g \in H$.

Using this we define the concept of a CSA group.

Definition 2.1. *A group G is a* **CSA-group** *or* **conjugately separated abelian group** *provided the maximal abelian subgroups are malnormal.*

Each CSA group must be CT. The converse however is not true in general.

Lemma 2.3. *The class of CSA groups is a proper subclass of the class of CT groups.*

Proof. We first show that every CSA-group is commutative transitive. Let G be a group in which maximal abelian subgroups are malnormal and suppose that M_1 and M_2 are maximal abelian subgroups in G with $z \neq 1$ lying in $M_1 \cap M_2$. Could we have $M_1 \neq M_2$? Suppose that $w \in M_1 \setminus M_2$. Then $w^{-1}zw = z$ is a non-trivial element of $w^{-1}M_2w \cap M_2$ so that

$w \in M_2$. This is impossible and therefore $M_1 \subset M_2$. By maximality we then get $M_1 = M_2$. Hence, G is commutative transitive whenever all maximal abelian subgroups are malnormal.

We now show that there do exist CT groups that are not CSA. In any non-abelian CSA-group the only abelian normal subgroup is the trivial subgroup 1. To see this suppose that N is any normal abelian subgroup of the non-abelian CSA-group G. Then N is contained in a maximal abelian subgroup M. Let $g \notin M$. Then

$$N = g^{-1}Ng \cap N \subset g^{-1}Mg \cap M.$$

The fact $N \neq 1$ would imply that $g \in M$ which is a contradiction.

Now let p and q be distinct primes with p a divisor of $q-1$. Let G be the non-abelian group of order pq . Then it is not difficult to prove that the centralizer of every non-trivial element of G is cyclic of order either p or q . Thus G is commutative transitive. However, the (necessarily unique) Sylow q-subgroup of G is normal in G. Hence from the argument above G cannot be CSA. $\square$

In the next section we give many more examples of CT but non CSA groups.

Although the class of CSA groups is a proper subclass of the CT groups, in the presence of full residual freeness (in fact even in the presence of just residual freeness) they are equivalent (see the next section). Fully residually free groups play a prominent role in the solution of the Tarski problems (see [FGMRS]). Finitely generated fully residually free groups are also known as **limit groups** since they arise (as initially observed by Sela [Se 1-5]) as limits of homomorphisms into free groups.

Definition 2.2. *A group G is* **residually free** *if for each non-trivial $g \in G$ there is a free group F_g and an epimorphism $h_g : G \to F_g$ such that $h_g(g) \neq 1$. Equivalently for each $g \in G$ there is a normal subgroup N_g such that G/N_g is free and $g \notin N_g$.*

The group G is **fully residually free** *provided to every finite set $S \subset G \setminus \{1\}$ of non-trivial elements of G there is a free group F_S and an epimorphism $h_S : G \to F_S$ such that $h_S(g) \neq 1$ for all $g \in S$.*

There is a beautiful theorem due independently to Gaglione and Spellman [GS] and Remeslennikov [Re] tying together full residual freeness, CT and and the property of being universally free which we will explain shortly.

In the 1960's G. Baumslag [GB] proved that a surface group is residually free, answering a question of Magnus. To do this he introduced what is now called **extensions of centralizers**. This concept became one of the main tools used by Kharlampovich, Myaasnikov and Remeslennikov in their structure theory of fully residually free groups and by Kharlampovich and Myasnikov in their solution to the Tarski problems. Using some of G.Baumslag's techniques, B. Baumslag proved [BB]

Theorem 2.4. *If G is residually free then the following are equivalent:*

(1) G is fully residually free

(2) G is CT

Gaglione and Spellman and independently Remeslennikov developed a truly amazing result that in some sense is the beginning of the solution of the Tarski problems. We first give some ideas from logic and model theory and then a brief introduction to the **Tarski problems**.

We start with a first-order language appropriate for group theory. This language, which we denote by L_0, is the first-order language with equality containing a binary operation symbol $\cdot$, a unary operation symbol $^{-1}$ and a constant symbol 1. A **sentence** in this language is a logical expression containing a string of variables $\overline{x} = (x_1, ..., x_n)$, the logical connectives $\vee, \wedge, \sim$ and the quantifiers $\forall, \exists$. A **universal sentence** of L_0 is one of the form $\forall \overline{x}\{\phi(\overline{x})\}$ where $\overline{x}$ is a tuple of distinct variables, $\phi(\overline{x})$ is a formula of L_0 containing no quantifiers and containing at most the variables of $\overline{x}$. Similarly an **existential sentence** is one of the form $\exists \overline{x}\{\phi(\overline{x})\}$ where $\overline{x}$ and $\phi(\overline{x})$ are as above.

If G is a group then the **universal theory** of G consists of the set of all universal sentences of L_0 true in G. We denote the universal theory of a group G by $Th_\forall(G)$. Since any universal sentence is equivalent to the negation of an existential sentence it follows that two groups have the same universal theory if and only if they have the same **existential theory**. The set of all sentences of L_0 true in G is called the **first-order theory** or the **elementary theory** of G. We denote this by $Th(G)$. We note that being **first-order** or **elementary** means that in the intended interpretation of any formula or sentence all of the variables (free or bound) are assumed to take on as values only individual group elements - never, for example, subsets of, nor functions on, the group in which they are interpreted. The **Tarski conjectures** or **Tarski problems**, solved independently by Kharlampovich and Myasnikov (see [KhM 1-5]) and Sela (see [Se 1-5]), say essentially that all countable nonabelian free groups have the same elementary theory. The following was well-known and much simpler.

Theorem 2.5. *All nonabelian free groups have the same universal theory.*

A **universally free group** G is a group that has the same universal theory as a nonabelian free group and, as we will see, all nonabelian finitely generated fully residually free groups are universally free.

One of the fundamental first steps in handling the Tarski problems was a theorem of Gaglione and Spellman [GS] and independently Remeslennikov [Re] who were able to extend the theorem of B. Baumslag to show that fully residually free is equivalent to universally free and that these (in the presence of residual freeness) are equivalent to both being CT and being CSA.

Theorem 2.6 (GS). *,[Re] If a nonabelian group G is residually free then the following are equivalent:*

(1) G is fully residually free

(2) G is CT

(3) G is CSA

(4) G is universally free.

Since this result a complete structure theory and algorithmic theory of the fully residually free groups has been developed. An important aspect of this development is that elements of fully residually free groups can be expressed as **infinite words** on a generating system. These infinite words can be manipulated and handled in an analogous manner to ordinary words in free groups (see [FGMRS]).

The positive solution to the Tarski Problems (see [KhM1-5],[Se 1-5] and [FGMRS]) is given in the next three theorems:

Theorem 2.7. *(Tarski 1) Any two nonabelian free groups are elementarily equivalent. That is any two nonabelian free groups satisfy exactly the same first-order theory.*

Theorem 2.8. *(Tarski 2) If the nonabelian free group H is a free factor in the free group G then the inclusion map $H \to G$ is an elementary embedding (see [FGMRS] for a precise definition of elementary embedding).*

The question also arises concerning the **decidability** of the theory of the nonabelian free groups, The **decidability** of the theory of nonabelian free groups means the question of whether there exists a recursive algorithm which, given a sentence ϕ of L_0, decides whether or not ϕ is true in every nonabelian free group. Kharlampovich and Myasnikov, in addition to proving the two above Tarski conjectures, also proved the following.

Theorem 2.9. *(Tarski 3) The elementary theory of the nonabelian free groups is decidable.*

Commutative transitivity becomes essential in building examples of fully residually free groups via the following construction.

Definition 2.3. *Let G be a CT group, let $u \in G \setminus \{1\}$ and let $M = Z_G(u)$ where $Z_G(u)$ is the centralizer of u in G. Suppose A is an abelian group. Then the group*

$$H = \langle G, A;\ rel\ (G),\ rel\ A, [A, z] = 1, \forall z \in M\rangle$$

is a **centralizer extension** *of G by A. If $A = \langle t\rangle$ is cyclic then $H = G(u, t)$ is the HNN extension*

$$G(u, t) = \langle G, t;\ rel\ (G), t^{-1}zt = z,\ for\ all\ z \in M\rangle$$

and is called the **free rank one extension of the centralizer** *M of u in G .*

Lemma 2.4. *(Baumslag, Myasnikov, Remeslennikov [BMR3]) Let G be a fully residually free group and A an abelian fully residually free group. Then a centralizer extension of G by A is again fully residually free.*

The proof of this result which is fundamental in all further considerations of fully residually free groups depends on the fact that the result can be reduced to free rank one extensions of centralizers and then on the following "big powers" argument (originally developed by G.Baumslag in [GB]). It is not hard to see that in a free group F if

$$b_0 t^{n_1} b_1 ... t^{n_k} b_k = 1$$

for infinitely many values of n_1, infinitely many values of $n_2, ...$, infinitely many values of n_k then t must commute with at least one of $b_0, ..., b_k$. Hence the family of homomorphisms $\phi_k : F(u, t) \to F$ from the rank one extension of the centralizer $C_F(u)$ into F, defined for every positive k by $\phi(t) = u^k$ and $\phi_k|_F = id$, is a discriminating family, as required.

The class of finitely generated fully residually free groups was introduced in a different direction by Sela in his proof of the Tarski problems. In Sela's approach these groups appear as limits of homomorphisms of a group G into a free group. In this guise they are called **limit groups**. Therefore a limit group is a finitely generated fully residually free group (see [FGMRS] for a proof of the equivalence of the two approaches.

As a by-product of the positive solution of the Tarski conjecture it was proved that the class of non-free groups that have exactly the same first order theory as the class of non-abelian free groups was nonempty. Such groups are called **elementary free groups** (or **elementarily free groups**) and both sets of authors provide complete characterizations of the finitely generated instances of them. In the Kharlampovich-Myasnikov approach these are the special NTQ-groups (see [KhM 1-5]). The primary examples of such groups are the orientable surface groups S_g of genus $g \geq 2$ and the nonorientable surface groups N_g of genus $g \geq 4$. That these groups are elementary free provides a powerful tool to prove some results in surface groups that are otherwise quite difficult. For example J.Howie [Ho] and independently O. Bogopolski [Bo], [BoS] proved that a theorem of Magnus about the normal closures of elements in free groups holds also in surface groups of appropriate genus. Their proofs were nontrivial. However it was proved (see [FGRS] and [GLS]) that this result is first order and hence automatically true in any elementary free group. In [FGRS] a large collection of such results was given. Such results were called *something for nothing results*. Of course any such first order result true in a nonabelian free group must hold in any elementary free group.

3 The Relationship Between CT and CSA

As mentioned, CSA always implies CT but the class of CSA groups is a proper subclass of the class of CT groups. In this section we prove that $PSL(2, K)$ is never CSA. However if K has characteristic 2 then $PSL(2, K)$ is always CT while $PSL(2, \mathbb{R})$ and $PSL(\mathbb{Q}_p)$ are also CT. The groups $PSL(2, K)$ for characteristic an odd prime p are never CT. $PSL(2, \mathbb{C})$ and more generally $PSL(2, K)$ where K is an algebraically closed field of characteristic 0 is never CT. Thus several of these types of groups provide an infinite number of example, both finite and infinite of CT non CSA groups. We also prove that a finite CSA group must be abelian. Wu [Wu] proves that there exist finite solvable CT groups for every solvability class. Hence the nonabelian ones provide more examples of CT non CSA groups.

We first consider some cases where CT and CSA are equivalent.

Lemma 3.1. *If G is residually free then $CT \cong CSA$.*

Proof. We assume Benjamin Baumslag's Theorem. CSA always imples CT so we assume that G is CT and show that it must be CSA. From Baumslag's theorem since G is residually free and CT it must be fully residually free so that we can assume that G is a fully residually free group with more than one element. Let $u \in G \setminus \{1\}$ and let M be its centralizer which we will denote by $C_G(u)$. Then M is maximal abelian in G. We claim that M is malnormal in G. If G is abelian, then $M = G$ and the conclusion follows trivially. Suppose that G is non-abelian. Suppose that $w = g^{-1}zg \neq 1$ lies in $g^{-1}Mg \cap M$. If $g \notin M$ then $[g, u] \neq 1$. Thus, there is a free group F and an epimorphism $h : G \to F, x \to \overline{x}$, such that $\overline{w} \neq 1$ and $[\overline{g}, \overline{u}] \neq 1$. Let $C = C_F(\overline{u})$. Then $\overline{w} \in \overline{g}^{-1}C\overline{g} \cap C$. However the maximal abelian subgroups in a free group are malnormal. This implies $\overline{g} \in C$, contradicting $[\overline{g}, \overline{u}] \neq 1$. This contradiction shows that $g^{-1}Mg \cap M \neq 1$ implies $g \in M$ and hence the maximal abelian subgroups in G are malnormal. ◻

Ciobanu, Fine and Rosenberger [CFR] generalized Benjamin Baumslag's theorem to what are called the class of $B\mathcal{X}$-groups. A class of groups $\mathcal{X}$ satisfies the property $B\mathcal{X}$ if a group G is fully residually $\mathcal{X}$ if and only if G is residually $\mathcal{X}$ and CT.

With this definition B. Baumslag's original theorem says that the class of free groups $\mathcal{F}$ satisfies $B\mathcal{F}$.

In [CFR] it was proved that a class of groups $\mathcal{X}$ satisfies $B\mathcal{X}$ under very mild conditions and hence the classes of groups for which this is true is quite extensive. In any class of groups satisfying $B\mathcal{X}$ the properties CT and CSA are equivalent.

Theorem 3.1. *(see [CFR]) Let $\mathcal{X}$ be a class of groups such that each nonabelian $H \in \mathcal{X}$ is CSA. Let G be a nonabelian and residually $\mathcal{X}$ group. Then the following are equivalent*

(1) G is fully residually $\mathcal{X}$

(2) G is CSA

(3) G is CT

Therefore the class $\mathcal{X}$ has the property $B\mathcal{X}$.

It follows that a class of groups $\mathcal{X}$ satisfies $B\mathcal{X}$ if each nonabelian $H \in \mathcal{X}$ is CSA. Examples of $B\mathcal{X}$ classes abound. In particular in [CFR] the following are listed.

Theorem 3.2. *Each of the following classes satisfies $B\mathcal{X}$:*

(1) The class of nonabelian free groups.

(2) The class of limit groups.

(3) The class of noncyclic torsion-free hyperbolic groups (see [GKM]).

(4) The class of noncyclic one-relator groups with only odd torsion (see [GKM]).

(5) The class of cocompact Fuchsian groups with only odd torsion.

(6) The class of noncyclic groups acting freely on Λ-trees where Λ is an ordered abelian group (see [CFR])

(7) The class of noncylic free products of cyclics with only odd torsion (see [GKM])

(8) The class of noncyclic torsion-free RG-groups (see[FMgrRR] and [CFR]).

(9) The class of conjugacy pinched one-relator groups of the following form

$$G = \langle F, t; tut^{-1} = v\rangle$$

where F is a free group of rank $n \geq 1$ and u, v are nontrivial elements of F that are not proper powers in F and for which $\langle u\rangle \cap x\langle v\rangle x^{-1} = \{1\}$ for all $x \in F$.

For the rest of this section we will concentrate on the situations where CT and CSA are not equivalent. That is we will examine CT non CSA groups. We need some prelimiaries. We saw that CT is given by a universal sentence and hence is captured by subgroups. The same is true for CSA

Lemma 3.2. *If G is a CSA group and $H \subset G$ then H is CSA.*

Proof. We first give a direct proof (see [GKM]). Let G be a CSA group and let H be a subgroup of G. Let A_H be a maximal abelian subgroup of H. We must show that A_H is malnormal in H. Let $x \in H$ with $xA_Hx^{-1} \cap A_H \neq \{1\}$. A_H is contained in a maximal

abelian subgroup A_G of G. Since G is CSA it follows that A_G is malnormal in G and so $x \in A_G$. Then $x \in (A_G \cap H) \subset A_H$ and hence A_H is malnormal in H.

The result also follows from the fact that CSA can also be described in terms of universal sentences. In particular the CSA property is described by the following pair of universal sentences.

$$(CT:)\forall x, y, z(((y \neq 1) \wedge ([x, y] = 1) \wedge ([y, z] = 1)) \rightarrow ([x, z] = 1))$$
$$(MAL:)\forall x, y, z((x \neq 1) \wedge (y \neq 1) \wedge ([x, y] = 1) \wedge ([x, z^{-1}yz] = 1) \rightarrow ([y, z] = 1))$$

□

Recall that the infinite dihedral group is the free product $D = \mathbb{Z}_2 \star \mathbb{Z}_2$. The group D then has the presentation $D = \langle x, y; x^2 = y^2 = 1 \rangle$. Then $xxyx^{-1} = yxyy^{-1} = yx = (xy)^{-1}$ and hence D is not CSA.

Lemma 3.3. *If the group G contains a subgroup isomorphic to the infinite dihedral group then G is not CSA.*

Corollary 3.1. *The modular group $M = PSL(2, \mathbb{Z})$ is not CSA.*

Proof. The modular group M is isomorphic to the free product $\mathbb{Z}_2 \star \mathbb{Z}_3$ of a cyclic group of order 2 and a cyclic group of order 3. Such a free product contains as a subgroup the free product $\mathbb{Z}_2 \star \mathbb{Z}_2$, a subgroup isomorphic to the infinite dihedral group. Therefore by Lemma 3.4 M cannot be CSA. □

Corollary 3.2. *The group $PSL(2, \mathbb{Q})$ where $\mathbb{Q}$ is the field of rational numbers is not CSA.*

Proof. $PSL(2, \mathbb{Q})$ contains M as a subgroup and hence cannot be CSA. □

Because of Wu Fen's work on finite CT groups (see Theorem 2.1) the groups $PSL(2, K)$, where K is a field, figure prominently in the analysis of CT and CSA groups. In [Wu] she also proves that there are finite solvable CT groups for every solvability class.

We prove the following two results. The first is that $PSL(2, K)$ for any field is never CSA.

Theorem 3.3. *Suppose that K is a field. Then the group $PSL(2, K)$ is not CSA.*

Proof. We consider the characteristic of K and handle each characteristic separately. If K is a field of characteristic $p \neq 2$ then the group $PSL(2, K)$ is not CT and hence it cannot be CSA.

Now let K be a field of characteristic 0. Then K contains a subfield isomorphic to $\mathbb{Q}$. Hence $PSL(2, K)$ contains a subgroup isomorphic to $PSL(2, \mathbb{Q})$. From Lemma 3.3 $PSL(2, \mathbb{Q})$ is not CSA and therefore $PSL(2, K)$ cannot be CSA.

Finally let K be a field of characteristic 2. Let $F = \mathbb{Z}_2$ be the two-element field. Then K contains a subfield isomorphic to F and hence $PSL(2, F) = PSL(2, \mathbb{Z}_2)$ is a subgroup of $PSL(2, K)$. However $PSL(2, \mathbb{Z}_2)$ is nonabelian of order 6 and hence is isomorphic to S_3 the symmetric group on 3 symbols. This group has an abelian normal subgroup or order 3 and hence is not CSA. It follows that $PSL(2, K)$ cannot be CSA. □

The next theorem handles the CT property for $PSL(2, K)$. It is more complex than for CSA.

Theorem 3.4. *Suppose that K is a field.*

(1) If char$(K) = 2$ then the group $PSL(2, K)$ is CT.

(2) If char$(K) = p$ where p is an odd prime the group $PSL(2, K)$ is not CT.

(3) If char$(K) = 0$ then the group $PSL(2, K)$ is CT if -1 is not a sum of two squares in K and not CT if -1 is a sum of two squares in K. In particular if $K = \mathbb{R}$, the real numbers or $K = \mathbb{Q}_p$ the p-adic numbers or any subfield of these then $PSL(2, K)$ is CT. On the other hand $PSL(2, \mathbb{C})$ is not CT and more generally $PSL(2, K)$ is not CT for any algebraically closed field of characteristic 0.

It follows from these two theorems that in the class of groups $PSL(2, K)$ there are infinitely many examples, both of finite order and infinite order of CT non CSA groups.

Proof. (Theorem 3.4) We do each characteristic separately with characteristic p the simplest.

Lemma 3.4. *If K is a field of characteristic $p \neq 2$ then the group $PSL(2, K)$ is not CT.*

Proof. From Wu's result a finite CT group must either be solvable or isomorphic to $PSL(2, K)$ where K is a field of characteristic 2. Hence if $p \neq 2$ we must have that $PSL(2, \mathbb{Z}_p)$ is not CT for the finite field $K = \mathbb{Z}_p$. If K is a field of characteristic $p \neq 2$ then $PSL(2, K)$ will contain $PSL(2, \mathbb{Z}_p)$ as a subgroup. Since the CT property is captured by subgroups it follows that $PSL(2, K)$ cannot be CT. □

Lemma 3.5. *If char$(K) = 0$ then the group $PSL(2, K)$ is CT if -1 is not a sum of two squares in K and not CT if -1 is a sum of two squares in K. In particular if $K = \mathbb{R}$, the real numbers or $K = \mathbb{Q}_p$ the p-adic numbers or any subfield of these then $PSL(2, K)$ is CT. Further $PSL(2, K)$ is CT for any real field. On the other hand $PSL(2, \mathbb{C})$ is not CT and more generally $PSL(2, K)$ is not CT for any algebraically closed field of characteristic* 0.

Proof. Suppose that K is a field with char$(K) = 0$ and suppose that there do not exist elements $x, y \in K$ such that $x^2 + y^2 = -1$. Let A, B, C be nontrivial elements of $PSL(2, K)$ with $AB = BA$ and $BC = CB$. Since K can be embedded in an algebraic closure k we may assume that each of A, B, C has one or two eigenvalues within k.

Case 1: B has one eigenvalue in k. Then this eigenvalue is already in K. After a suitable conjugation in $PSL(2, K)$ we may assume that

$$B = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } PSL(2, K)$$

From $AB = BA$ we get that either $c = 0$ or $a = d = -b, -c = 2a$. We must have $c = 0$ for otherwise this implies that

$$1 = ad - bc = a^2 - 2a^2 = -a^2 - 0^2$$

and hence -1 is a sum of two squares in K contrary to assumption. Further $AB = BA$ then leads to $a = d = \pm 1$. Hence A has the form

$$A = \pm \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \text{ in } PSL(2, K).$$

An analogous statement holds for C since $BC = CB$. Therefore C has this form also and hence $AC = CA$ in Case 1.

Case 2: B has two eigenvalues in k. After a suitable conjugation in $PSL(2, k)$ we may assume that

$$B = \pm \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in PSL(2, k).$$

Since B is nontrivial we have $\alpha \neq \pm 1$.

Let

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } PSL(2, k).$$

Then from $AB = BA$ we get that either $b = c = 0$ or $c \neq 0, c\alpha = -c\alpha^{-1}$ or $b \neq 0, b\alpha = -\alpha^{-1}b$.

If $c \neq 0$ or $b \neq 0$ then $\alpha = -\alpha^{-1}$ and hence the trace, $tr(B) = 0$.

Analogously for C. Let

$$C = \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix} \text{ in } PSL(2,k).$$

Then from $BC = CB$ we get that either $f = g = 0$ or $tr(B) = 0$.

If $b = c = f = g = 0$ then $AC = CA$ and hence here in case 2 we may assume that $tr(B) = 0$.

We now consider $A, B, C \in PSL(2, K)$ with $tr(B) = 0$. Let

$$B = \pm \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

We have that $\gamma \neq 0$ for if $\gamma = 0$ then $-\alpha^2 = -\alpha^2 + 0^2 = 1$ contrary to assumption that -1 is not a sum of squares.

Hence by conjugation we may assume that B has the form

$$B = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then from $AB = BA$ we get that either

$$A = \pm \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \text{ or } A = \pm \begin{pmatrix} -x & y \\ y & x \end{pmatrix}.$$

If

$$A = \pm \begin{pmatrix} -x & y \\ y & x \end{pmatrix}$$

then

$$-x^2 - y^2 = 1$$

contradicting the assumption on -1 not being a sum of squares. Therefore

$$A = \pm \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

Analogously let
$$C = \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$
Then from $BC = CB$ we get that
$$C = \pm \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$$
But in this case $AC = CA$.

Therefore altogether $PSL(2, K)$ is CT if -1 is not a sum of two squares in K.

Now suppose that $-1 = x^2 + y^2$ in K. Since K has characteristic 0 we have $\mathbb{Q} \subset K$. Let α, β be nonzero elements of $\mathbb{Q}$ such that $\alpha^2 + \beta^2 = 1$. For example let $\alpha = \frac{3}{5}, \beta = \frac{4}{5}$. Now let
$$A = \pm \begin{pmatrix} -x & y \\ y & x \end{pmatrix}$$
$$B = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
and
$$C = \pm \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$
Then A, B, C are three nontrivial elements of $PSL(2, K)$ with $AB = BA$, $BC = CB$ but $AC \neq CA$ so the group is not CT.

Notice that if -1 is itself a square in K the group $PSL(2, K)$ then cannot be CT. In particular this is true for the complex numbers $\mathbb{C}$ and more generally for any algebraically closed field K. We give an example in $PSL(2, \mathbb{C})$ to clarify this.

In $PSL(2, \mathbb{C})$ we have the projective matrices
$$T = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, U = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, V = \pm \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \alpha \neq \pm 1.$$
As linear fractional transformations these are
$$T : z' = -\frac{1}{z}, U : z' = -z, V : z' = \alpha^2 z.$$
By a direct computation U commutes with T and V but T and V do not commute. Therefore $PSL(2, \mathbb{C})$ is not CT.

Exactly the analogous example works in any field K of characteristic 0 where -1 is a square. Therefore the example holds in $PSL(2, K)$ for any algebraically closed field of characteristic 0.

Notice further that the lemma applies to $PSL(2, \mathbb{R})$ for the real numbers $\mathbb{R}$ and for any subfield of $\mathbb{R}$, in particular any algebraic number field, and for any subgroup of $PSL(2, \mathbb{R})$. Hence any Fuchsian group is CT.

In general, fields where -1 is not a sum of squares are called **real fields** and have been extensively studied (see [La]). If -1 is not a sum of squares then it is not a sum of two squares and hence $PSL(2, K)$ is CT for any real field. $\square$

Lemma 3.6. *Let K be a field of characteristic 2. Then $PSL(2, K)$ is CT.*

Proof. Let K be a field of characteristic 2 and let $F = \mathbb{Z}_2$ be the two element field. Clearly F is a subfield of K. For a field of characteristic 2 we have $PSL(2, K) = SL(2, K)$ so we show that $SL(2, K)$ is CT. Now $PSL(2, F) \cong S_3$, the symmetric group on three symbols. This group is CT so we now may assume that $|K| \geq 4$.

Let $\overline{K}$ be an algebraic closure of K and let k be the algebraic closure of F in $\overline{K}$. Then we have the tower of fields.

$$\begin{tikzcd}[column sep=small]
 & \overline{K} \arrow[dl, dash] \arrow[dr, dash] & \\
k \arrow[dr, dash] & & K \arrow[dl, dash] \\
 & F &
\end{tikzcd}$$

Tower of Fields 1

From [Hod, p. 47] we have that one form of Hilbert's Nullstellensatz says that if A is an algebraically closed field and E is a finite systems of equations and inequations with coefficients from A, such that some field extending A contains a solution of E, then A already contains a solution of E (see also Jacobson [Ja], p. 425). It follows from this that an existentially closed field (see[Ja] for a definition) is the same thing as an algebraically closed field. We now use a bit of model theory. We refer the reader to [BeS] or [FGMRS] for a discussion of ultrapowers.

Since k and $\overline{K}$ are algebraically closed they are existentially closed and hence, since $k \subset \overline{K}$ we must have the universal equivalence $k \equiv_{\forall} \overline{K}$.

By Lemma 3.8 in [BeS] (p. 187) the field $\overline{K}$ embeds in an ultrapower ${}^{\star}k = k^I/D$ of k. We now have the diagram;

$$\begin{tikzcd}
& {}^{\star}k \arrow[d, dash] & \\
& \overline{K} \arrow[dl, dash] \arrow[dr, dash] & \\
k \arrow[dr, dash] & & K \arrow[dl, dash] \\
& F &
\end{tikzcd}$$

Tower of Fields 2

For a field E making the obvious identification of ${}^{\star}SL(2, E) = SL(2, E)^I/D$ with $SL(2, {}^{\star}E) = SL(2, E^I/D)$ we have the diagram of group inclusions.

$$\begin{tikzcd}
& {}^{\star}SL(2, k) \arrow[dl, dash] \arrow[dr, dash] & \\
SL(2, k) & & SL(2, K)
\end{tikzcd}$$

Group Inclusions

From the result of Wu we have that $SL(2, k)$ is CT since k is locally finite. Further ${}^{\star}SL(2, k) \equiv_{\forall} SL(2, k)$ and hence ${}^{\star}SL(2, k)$ is CT since CT is expressed as a universal sentence in the language L_0 of group theory. But the CT property is inherited by subgroups and therefore it follows that $SL(2, K)$ is CT. $\square$

These three lemmas complete the proof of Theorem 3.1.

$\square$

We now prove:

Theorem 3.5. *Let G be a finite CSA group. Then G is abelian.*

Proof. Let G be a finite CSA group. Since CSA implies CT we then have G is a finite CT group. From Wu's theorem G is then either solvable or isomorphic to $PSL(2, K)$ for a finite field of characteristic 2. If $G \cong PSL(2, K)$ then from Theorem 3.2 G cannot be CSA. Hence G must be solvable. If the solvability class is $d > 1$ then the element of the derived series G^{d-1} is a nontrivial abelian normal subgroup and hence G cannot be CSA in this case. It follows that the solvability class must be $d = 1$ and therefore G is abelian. $\square$

4 Infinite CT non CSA Groups

For finite groups a CT group is either solvable or simple. The situation for infinite CT groups is more From Theorem 3.3 and Wu's analysis we have that finite CSA groups are abelian. Wu proves that there exist finite CT groups of every solvability class and hence the nonabelian ones (those of solvability class $d > 1$ provide examples of finite CT non CSA groups. The situation for infinite CT groups is more complicated. First notice that the infinite Tarksi simple groups mentioned in previous section are both CT and CSA while the free product of two finite CT non CSA groups such as $S_3 \star S_3$ is an infinite CT but non CSA group.

Lemma 4.1. *Let G, H be any two CT non CSA groups. Then the free product $G \star H$ is also CT non CSA.*

Proof. The free product of CT groups is again CT so $G \star H$ is CT. However G can be considered as a subgroup of the free product so $G \star H$ cannot be CSA. □

We now give several results characterizing infinite CT non CSA groups. Notice that if a group G contains a nontrivial abelian normal subgroup then it cannot be CSA. This is almost enough to characterize when a CT group is not CSA.

Theorem 4.1. *A CT group G is not CSA if and only if G contains a nonabelian subgroup G_0 which itself contains a nontrivial abelian subgroup H which is normal in G_0*

Proof. Suppose that G contains a nonabelian subgroup G_0 which itself contains a nontrivial abelian subgroup H which is normal in G_0. Then G_0 cannot be CSA and hence G cannot be CSA.

Conversely suppose that G is CT but not CSA. Recall that in the presence of the group axioms the CSA property is captured by the pair of universal sentences

$$(CT:)\forall x, y, z(((y \neq 1) \wedge ([x, y] = 1) \wedge ([y, z] = 1)) \to ([x, z] = 1))$$

$$(MAL:)\forall x, y, z((x \neq 1) \wedge (y \neq 1) \wedge ([x, y] = 1) \wedge ([x, z^{-1}yz] = 1) \to ([y, z] = 1))$$

It follows that being CT but not CSA is captured (in the presence of the group axioms) by

$$(CT:)\forall x, y, z(((y \neq 1) \wedge ([x, y] = 1) \wedge ([y, z] = 1) \to ([x, z] = 1))$$

$$(NOTMAL:)\exists x, y, z((x \neq 1) \wedge (y \neq 1) \wedge ([x, y] = 1) \wedge ([x, z^{-1}yz] = 1) \wedge ([y, z] \neq 1))$$

Now suppose that G is CT but not CSA. Let $g, h, k \in G$ such that if $g = x, h = y, z = k$ then these three elements verify NOTMAL in G. Consider the subgroup $G_0 = \langle g, h, k \rangle$ of G. This is nonabelian since $[h, k] \neq 1$.

Now consider the subgroup $A = \langle h \rangle^{G_0}$, the normal closure of $\langle h \rangle$ in G_0. Since $h \neq 1$ the subgroup A is nontrivial. We claim that A is abelian which will complete the proof.

Now g and h commute with h and further $g \neq 1$ commutes with $k^{-1}hk$ and commutes with h so by CT, $k^{-1}hk$ commutes with h. From the fact that $k^{-1}hk$ commutes with h we get that $k(k^{-1}hk)k^{-1}$ commutes with khk^{-1} and so khk^{-1} commutes with h. Hence if $u \in \{g, g^{-1}, h, h^{-1}, k, k^{-1}\}$ then $uh^{\epsilon}u^{-1}$ commutes with h where $\epsilon = \pm 1$. It follows that these conjugates commute with each other.

If A were not abelian there would be a word $w(x, y, z)$ of shortest length such that

$$w(g, h, k)^{-1} h w(g, h, k)$$

did not commute with h. Choose such a w with $w = v(x, y, z)u$ and $u \in \{g, g^{-1}, h, h^{-1}, k, k^{-1}\}$. Then by minimality $v(g, h, k)^{-1} h v(g, h, k)$ commutes with h. Let $\overline{u}$ be the value of u in G_0 so that $\overline{u} \in \{g, g^{-1}, h, h^{-1}, k, k^{-1}\}$. From $v(g, h, k)^{-1} h v(g, h, k)$ commuting with h we get that $\overline{u}^{-1} v(g, h, k)^{-1} h v(g, h, k) \overline{u}$ commutes with $\overline{u}^{-1} h \overline{u}$. But $\overline{u}^{-1} h \overline{u}$ commutes with h so by CT

$$w(g, h, l)^{-1} h w(g, h, k) = \overline{u}^{-1} v(g, h, k)^{-1} h v(g, h, k) \overline{u}$$

commutes with h contradicting our choice of $w(x, y, z)$. Therefore this contradiction shows that A must be abelian. ☐

To proceed further we need the following concept. A group G is **monolithic** if G contains a unique nontrivial minimal normal subgroup N (see [N]). This subgroup is then called the **monolith**. Our first result is the following:

Theorem 4.2. *Let G be a nontrivial CT group which contains no nontrivial abelian normal subgroup. If G has a composition series then G is monolithic whose monolith N is a simple nonabelian CT group.*

Proof. Notice that if the monolith $N \cong PSL(2, K)$ for K a field of characteristic 2, which is the situation for finite CT groups with no abelian normal subgroups then G would not be CSA. However as pointed out above there do exists simple nonabelian CT groups that are CSA.

If H is a group then a descending chain of subgroups

$$H = H_0 \supset H_1 \supset \cdots H_n$$

is a **chief series** (see [Hal] P. 124) from H to H_n provided H_i is normal in H for all $i = 0, 1, ..., n$ and for all $i = 0, ..., n$, H_i is maximal normal in H_{i-1}.

Let G be a CT group with a composition series and no abelian normal subgroup. Since G is assumed to have a composition series it follows from Theorem 8.6.1 ([Ha], p.131) that G has a chief series

$$G = G_0 \supset G_1 \supset \cdots \supset G_{n-1} \supset G_n = \{1\}.$$

Let $M = G_{n-1}$. Then $G_n = \{1\}$ is maximal normal in M and hence there is no subgroup N normal in G such that $M \supset N \supset \{1\}$. It follows that M is a minimal normal subgroup in G.

We claim that M is unique. Suppose that M_1 and M_2 are minimal normal subgroups of G with $M_1 \neq M_2$. By assumption neither is abelian. By minimality $M_1 \cap M_2 = \{1\}$. It follows that the subgroup $H = \langle M_1, M_2 \rangle$ generated by M_1, M_2 is their direct product. That is $H = M_1 \times M_2$. However a direct product of nonabelian groups is not CT a contradiction since G is CT and CT is inherited by subgroups. Therefore M is a unique minimal normal subgroup and hence G is monolithic with monolith M.

Again from Theorem 8.6.1 in [Ha] M is a direct power A^m of a simple group A. Since G contains no normal abelian subgroup it follows that A^m is not abelian and hence A is not abelian. If $m > 1$ the monolith A^m is not CT, again a contradiction and therefore $m = 1$ and the monolith is a nonabelian simple CT group. $\square$

We now consider monolithic groups with monolith isomorphic to $PSL(2, K)$ for a field K of characteristic 2.

Theorem 4.3. *Let G be a monolithic group with monolith isomorphic to $PSL(2, K)$ where K is a field of characteristic 2 with $|K| \geq 4$. Then if G is CT we must have $G \cong PSL(2, K)$ and hence G is non CSA.*

We need two preliminary results before we prove this theorem.

Lemma 4.2. *Let G be a monolithic group with monolith M isomorphic to $PSL(2, K)$ where K is a field of characteristic 2. If G is CT, G then embeds into $Aut(M)$.*

Proof. Since M is normal in G we get a map ϕ from G to $\mathrm{Aut}(M)$ by mapping $g \in G$ to conjugation on M by g. Now $M = SL(2, K)$ is nonabelian. Choose $a, b \in M$ such that $ab \neq ba$ and suppose that $z \in ker(\phi)$. Then $zaz^{-1} = a$ and $zbz^{-1}z = b$. Now z commutes with both a and b. If $z \neq 1$ and $ab \neq ba$ this contradicts the assumption that G is CT. Hence $ker(\phi) = \{1\}$ and hence ϕ is an embedding. $\square$

Recall that an algebraic structure is **rigid** if it admits only the identity automorphism.

Theorem 4.4. *Let K be a field of characteristic 2 with $|K| \geq 4$. Then K is not rigid.*

Proof. Let K be a field of characteristic 2 with $|K| \geq 4$. Then the map $\sigma : K \to K$ given by $\sigma(x) = x^2$ for all $x \in K$ is an injective homomorphism. If it is surjective we are done since the only roots of $x^2 - x$ over K are 0 or 1 and we thus get a nontrivial automorphism.

Assume now that K contains an element θ which is not a square in K and assume that K is rigid. Now consider the simple group $M = SL(2, K)$. Recall that $SL(2, K) = PSL(2, K)$. By Theorem 15.3.2 in [Sc] we have that $\mathrm{Aut}(M)$ is complete. Since we assumed that K is rigid $\mathrm{Aut}(M)$ consists solely of inner automorphisms and hence M is isomorphic to $\mathrm{Aut}(M)$ and hence M is complete. By Theorem 15.3.1 in [Sc] if H is any overgroup of M in which M is normal then H has the internal direct product representation $M \times C_H(M)$ where $C_H(M)$ is the centralier of H in M.

Thus since $M = SL(2, K)$ is normal in $H = GL(2, K)$, H has the internal direct product representation $H = M \times C_H(M)$. We claim that the centralizer of M in H consists of the nonzero scalar matrices. It is straightforward that any matrix in H which commutes with both

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

must be scalar.

Now fix an element $\theta \in K$ which is not a square in K and let

$$C = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$$

Then C has a unique representation of the form AB where $A \in M$ and $B \in C_H(M)$.

Since $B \in C_H(M)$ the matrix for B is a scalar matrix $\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$ for some nonzero $\beta \in K$. But then

$$\theta = \det \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix} = \det(A)\det(B) = \beta^2.$$

Then θ is a square in K contrary to assumption. It follows that K cannot be rigid. ☐

We now give the proof of Theorem 4.2.

Proof. (Theorem 4.2) Let G be a monolithic group with monolith isomorphic to $PSL(2, K)$ where K is a field of characteristic 2 and $|K| \geq 4$ and suppose that G is CT.

From Theorem 4.3 we have that $PSL(2, K)$ is not rigid. From [Wan] we have that an automorphism of $SL(2, K)$ is of the form $A \mapsto PA^\sigma P^{-1}$ or $A \mapsto P(A^\iota)^t P^{-1}$ where σ is an

automorphism of K, ι is an antiautomorphism of K and A^t is the transpose of A. Since K is commutative being a field any anti-automorphism is already an automorphism. Further the transpose operator is an anti-automorphism but not an automorphism of $SL(2, K)$. It follows than that the maps of the form $A \mapsto P(A^\iota)^t P^{-1}$ do not occur here.

Since K is not rigid there is a nonidentity autmorphism of K. Let σ be such a nonidentity automorphism and let ρ be the nontrivial automorphism of $PSL(2, K)$ given by

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}.$$

Now let A, B be the elements of $SL(2, K)$ given by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let α denote conjugation in $SL(2, K)$ by A and β denote conjugation in $SL(2, K)$ by B.

By direct computation we have that in Aut($SL(2, K)$ the automorphisms α, β both commute with τ, that is $\alpha\tau = \tau\alpha$ and $\tau\beta = \beta\tau$. However again by direct computation we have $\alpha\beta \neq \beta\alpha$.

By Lemma 4.1 G embeds in Aut(M) =Aut($SL(2, K)$ and it follows that the image of G in the automorphism group must also be CT. This combined with the computations above imply that no transformation of the form

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}$$

for a nontrivial automorphism σ of K can occur in the image of G in Aut(M). Thus the image of G in Aut(M) must consist solely of inner automorphisms of M. Hence for every $g \in G$ there exists an $h \in M$ such that $gxg^{-1} = hxh^{-1}$ for all $x \in M$. Therefore

$$g^{-1}h \in C_G(M) = ker(G \to \text{Aut}(M)) = \{1\}.$$

Thus $g = h \in M$ and since g, h were arbitrary $G = M$ completing the proof. $\square$

Finally recall that a class of groups $\mathcal{X}$ is **axiomatic** is this class is defined in terms of a set of first order sentences (see [FGMRS]) or axioms. We have seen that the CT property is given by the group axioms together with

$$(CT:)\forall x, y, z(((y \neq 1) \wedge ([x, y] = 1) \wedge ([y, z] = 1) \rightarrow ([x, z] = 1))$$

while the class of CSA groups is captured by

$$(CT:)\forall x,y,z(((y\neq 1)\wedge([x,y]=1)\wedge([y,z]=1)\rightarrow([x,z]=1))$$

$$(MAL:)\forall x,y,z((x\neq 1)\wedge(y\neq 1)\wedge([x,y]=1)\wedge([x,z^{-1}yz]=1)\rightarrow([y,z]=1)).$$

Finally being CT but not CSA is captured (in the presence of the group axioms) by

$$(CT:)\forall x,y,z(((y\neq 1)\wedge([x,y]=1)\wedge([y,z]=1)\rightarrow([x,z]=1))$$

$$(NOTMAL:)\exists x,y,z((x\neq 1)\wedge(y\neq 1)\wedge([x,y]=1)\wedge([x,z^{-1}yz]=1)\wedge([y,z]\neq 1)).$$

If $\mathcal{G}$ represent the class of CT groups, $\mathcal{H}$ the class of CSA groups and $\mathcal{M}=\mathcal{G}\cap(\mathcal{H})^{\rfloor}$ the class of CT but not CSA groups, then all three classes are axiomatic.

Theorem 4.5. *The class $\mathcal{G}$ of CT groups, the class $\mathcal{H}$ of CSA groups and the class $\mathcal{M}=\mathcal{G}\cap(\mathcal{H})^{\rfloor}$ of CT non CSA groups are all axiomatic.*

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